

**OPTIMAL LAYOUT OF EMBEDDING GENERALIZED
PETERSEN GRAPH $P(n, m)$ INTO CYCLE OF TRIANGLE**

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Abstract: Graph embedding is an important tool in parallel computation and interconnection networks. Interconnection network plays a major role in parallel processing and computation system. The embedding technique helps in demonstrating parallel computer and the interconnection network is given through guest graph where the challenge is to find out non-verlapping minimum wirelength. In this paper with Petersen graph as guest graph and cycle of triangle as host graph, we find the exact wirelength of embedding Petersen graph $P(n, m)$ into cycle of triangle.

Keywords and Phrases: Embedding, edge congestion, Petersen graph, cycle of triangle, wirelength.

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1. Introduction and Preliminaries

Graph embedding is an important tool in parallel computation or simulation of different interconnection networks. Using graph theory, parallel computer is modelled with vertices and edges denoting processes and communication between the processes respectively. Applications of graph embedding is rooted from VLSI designs, data structures and data representation, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture and structural engineering and so on [9, 12].

Graph embedding has its own dominating role in understanding the architectures of nature and human. We have varieties of results proved in circular wirelength of generalized Petersen graph [10], Petersen graph into certain trees [9], hypercubes into grids [2], binary trees into hypercubes [5], hypercubes into cylinder, snakes and caterpillars [7], hypercubes on a grid [8] and so on. In this paper, we will find out the exact wirelength of embedding Petersen graph $P(n, m)$ into cycle of triangle CT_n using generalized congestion lemma, congestion lemma, partition lemma and k -partition lemma.

Definition 1.1. [2] Let G and H be finite graphs with n vertices. Let $V(G)$ and $V(H)$ denote the vertex sets of G and H respectively. Let $E(G)$ and $E(H)$ denote the edge sets of G and H respectively. An embedding f of G into H is defined as follows:

- (i) f is an injective map from $V(G) \rightarrow V(H)$,
- (ii) P_f is an injective map from $E(G)$ to $P_f(u, v) : P_f(u, v)$ is a path in H between $f(u)$ and $f(v)$.

Definition 1.2. [2] The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H . Let $EC_f(G, H(e))$ denote the number of edges (u, v) of G such that e is in the path $P_f(u, v)$ between $f(u)$ and $f(v)$ in H . In other words, $EC_f(G, H(e)) = |(u, v) \in E(G) : e \in P_f(u, v)|$ where $P_f(u, v)$ denotes the path between $f(u)$ and $f(v)$ in H with respect to f .

Definition 1.3. [8] The wirelength of an embedding f of G into H is given by $WL_f(G, H) = \sum_{(u,v) \in E(G)} d_H(f(u), f(v)) = \sum_{e \in E(G)} EC_f(G, H(e))$ where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in H . Then the wirelength of G into H is defined as $WL(G, H) = \min WL_f(G, H)$.

In the literature [3], there are two problems have been considered with respect to edge isoperimetric problem. The first problem is to find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality and the second one is to find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. That is, given $X \subseteq V(G)$, we denote $\Theta_G(X) = \{uv \in E(G) : u \in X, v \notin X\}$ and $I_G(X) = \{uv \in E(G) : u, v \in X\}$. For $k = 1, 2, \dots, n$, let $\Theta_G(k) = \min_{X \subseteq V(G), |X|=k} |\Theta_G(X)|$, and $I_G(k) = \max_{X \subseteq V(G), |X|=k} |I_G(X)|$. The above two problems are equivalent in the case of regular graphs and are NP-complete in general [6].

Lemma 1.1. (Congestion Lemma) [8] *Let G be an r regular graph and f be an embedding of G into H . Let S be an edge cut of H such that the removal of edges of S leaves H into two components H_1 and H_2 and $G_1 = G[f^{-1}(H_1)]$ and $G_2 = G[f^{-1}(H_2)]$. Also S satisfies the following conditions:*

- (i) *For every edge $a, b \in G_i$, $i = 1, 2$, $P_f(a, b)$ has no edges in S .*
- (ii) *For every edge $(a, b) \in G$ with $a \in G_1$ and $b \in G_2$, $P_f(a, b)$ has exactly one edge in S .*
- (iii) *G_1 is maximum subgraph on k vertices where $k = |V(G_1)|$.*

Then $EC_f(S)$ is minimum and $EC_f(S) = rk - 2|E(G_1)|$.

Lemma 1.2. (Generalized Congestion Lemma) [10] *Let f be an embedding of G into H . Let S be an edge cut of H such that the removal of edges of S splits H into k components H_i , $1 \leq i \leq k$. Let $G_i = G[f^{-1}(H_i)]$, $1 \leq i \leq k$, be such that the sets G_i are optimal and S satisfies the following conditions:*

- (i) *For every edge $(u, v) \in G_i$, $1 \leq i \leq k$, $P_f(u, v)$ has no edges in S .*
- (ii) *For every edge $(u, v) \in G$ with $u \in G_i$ and $v \in G_j$ for $i < j$, $P_f(u, v)$ has exactly one edge in S .*

Then $EC_f(S)$ is minimum over all possible embeddings and $EC_f(S) = \frac{1}{2} \sum_{i=1}^k \Theta_G(m_i)$ where $m_i = |V(G_i)|$. Further when G is an r -regular graph, $EC_f(S) = \frac{r}{2} |V(G)| - \sum_{i=1}^k |E(G_i)|$.

Lemma 1.3. (Partition Lemma) [8] *Let $f : G \rightarrow H$ be an embedding. Let $\{S_1, S_2, \dots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H . Then, $WL_f(G, H) = \sum_{i=1}^p EC_f(S_i)$.*

Lemma 1.4. (k -Partition Lemma) [10] *Let $f : G \rightarrow H$ be an embedding. Let $[kE(H)]$ denote a collection of edges of H with each edge in H repeated exactly k times. Let $\{S_1, S_2, \dots, S_p\}$ be a partition of $[kE(H)]$ such that each S_i is an edge cut of H . Then $WL_f(G, H) = \frac{1}{k} \sum_{i=1}^p EC_f(S_i)$.*

2. Generalized Petersen Graph $P(n, m)$

Definition 2.1. [10] *The generalized Petersen graph $P(n, m)$, $1 \leq m \leq n - 1$ and $n \neq 2m$, consists of an outer n -cycle $\{u_1, u_2, \dots, u_n\}$, a set of n spokes (u_i, v_i) , $1 \leq i \leq n$ and n inner edges (v_i, v_{i+m}) with indices taken modulo n . It is a 3-regular graph and contains $2n$ vertices and $3n$ edges.*

Parallel Labeling: *For $1 \leq i \leq n$, we call the vertices u_i and v_i of $P(m, n)$ as*

outer rim and inner rim vertices respectively and label the vertices u_i and v_i as $2i - 2$ and $2i - 1$ respectively. We call this labeling as parallel labeling [10] of the generalized Petersen graph $P(n, m)$.

Definition 2.2. Let C_n be a cycle on n vertices with vertex set $\{v_i : 0 \leq i \leq n-1\}$. The cycle of triangle CT_n is obtained from C_n by adding new vertices $\{u_i : 0 \leq i \leq n-1\}$ such that v_i and $v_{(i+1)}$ are joined to u_i , $0 \leq i \leq n-1$, where the suffix is taken modulo n .

Zigzag Labeling: For $0 \leq i \leq n-1$, we label the vertices u_i as $2i$ and v_i , $0 \leq i \leq n-1$ as $2i-1$ and v_0 as $2n-1$. We call this labeling as zigzag labeling.

Theorem 2.1. [10] The number of edges in the subgraph induced by any set of k vertices of $P(n, 1)$, $5 \leq k \leq n$ is atmost $k + \lfloor \frac{k}{2} \rfloor - 2$.

Theorem 2.2. The number of edges in the subgraph induced by any set of k vertices of $P(n, 2)$, $6 \leq k \leq n$ is atmost $k + \lfloor \frac{k}{2} \rfloor - 3$.

Proof. Let $U = \{v_1, v_2, \dots, v_k\}$ be a set of k vertices of $P(n, 2)$, not necessarily with consecutive labels. Let H be the subgraph of $P(n, 2)$ induced by the vertices of U .

Case 1: If all the vertices of H are outer rim vertices then $|E(H)| \leq k$. Hence $|E(H)| \leq k + \lfloor \frac{k}{2} \rfloor - 3$. The same argument is true if all the vertices of H are inner rim vertices.

Case 2 : Let k_1 and k_2 be the number of outer and inner rim vertices of H respectively such that $k = k_1 + k_2$. Without loss of generality, we prove the result for $k_1 \geq \lceil \frac{k}{2} \rceil$ and $k_2 \leq \lfloor \frac{k}{2} \rfloor$.

Subcase 2.1: $(k_2 < \lfloor \frac{k}{2} \rfloor)$: If outer rim vertices induce one or more arcs then the number edges induced by these vertices is atmost $k_1 - 1$. But $E(H)$ can have a maximum of k_2 spokes and $k_2 - 1$ edges induced by the inner rim vertices. We have $k_2 \leq \lfloor \frac{k}{2} \rfloor - 1$ and hence $|E(H)| \leq k_1 - 1 + k_2 + k_2 - 1 = k + k_2 - 2 \leq k + \lfloor \frac{k}{2} \rfloor - 3$.

Subcase 2.2: $(k_2 = \lfloor \frac{k}{2} \rfloor)$: In this case $k_1 = k_2$ if k is even, and $k_1 = k_2 + 1$ if k is odd. Assume that outer rim vertices induce an arc. Then the number of edges induced by outer rim vertices is $k_1 - 1$. If $E(H)$ contains k_2 spokes then k_2 inner rim vertices induce $k_2 - 2$ edges of H and hence $|E(H)| = k_1 - 1 + k_2 + k_2 - 2 = k + \lfloor \frac{k}{2} \rfloor - 3$. Suppose the number of spokes in $E(H)$ is less than k_2 . If $E(H)$ contains a cycle on $\lfloor \frac{k}{2} \rfloor$ inner rim vertices then k must be even and $k = n$. Since $\lceil \frac{k}{2} \rceil$ number of outer rim vertices induce an arc, we get the number of spokes in $E(H)$ is atmost $\lceil \frac{k}{4} \rceil$. Therefore $|E(H)| < k_1 - 1 + k_2 + \lceil \frac{k}{4} \rceil \leq k - 2 + \lceil \frac{k}{4} \rceil$. Since $k \geq 6$, we get $\lceil \frac{k}{4} \rceil \geq 2$ and $\lfloor \frac{k}{2} \rfloor \geq 3$. Thus $|E(H)| \leq k + \lfloor \frac{k}{2} \rfloor - 3$. If $E(H)$ contains no cycle on $\lfloor \frac{k}{2} \rfloor$ inner rim vertices then it contains one or more arcs induced by inner rim vertices.

Then the number of edges induced by inner rim vertices atmost $k_2 - 1$. Therefore $|E(H)| < k_1 - 1 + k_2 + k_2 - 1 \leq k + \lfloor \frac{k}{2} \rfloor - 3$.

Assume that outer rim vertices induce more than one arc then the number edges induced by these vertices is atmost $k_1 - 2$. Suppose the number of spokes in $E(H)$ is less than k_2 . Then the number of edges induced by inner rim vertices is atmost $k_2 - 1$. Hence $|E(H)| < k_1 - 2 + k_2 + k_2 - 1 < k + \lfloor \frac{k}{2} \rfloor - 3$. Suppose the number of spokes in $E(H)$ is k_2 . If $E(H)$ contains a cycle on $\lfloor \frac{k}{2} \rfloor$ inner rim vertices then k must be even and $k = n$. In this case the number edges induced by outer rim vertices is zero. Therefore $|E(H)| = k_2 + k_2 \leq k$. If $E(H)$ contains no cycle on $\lfloor \frac{k}{2} \rfloor$ inner rim vertices then it contains one or more arcs induced by inner rim vertices. Then the number of edges induced by inner rim vertices is atmost $k_2 - 1$. Therefore $|E(H)| \leq k_1 - 2 + k_2 + k_2 - 1 \leq k + \lfloor \frac{k}{2} \rfloor - 3$.

Theorem 2.3. *Let H be a subgraph of $P(n, 2)$ induced by k vertices, $6 \leq k \leq n$ such that*

- (i) *if k is even, the labels of the k vertices are $\{i + 1, i + 2, \dots, i + k\}$ and*
- (ii) *if k is odd, the labels of the $k - 1$ vertices are $\{i + 1, i + 2, \dots, i + k - 1\}$ and k^{th} vertex is of any one of the labels $i - 1, i, i + k$ or $i + k + 1$.*

where i is odd and the labels are taken modulo $2n$. Then H is a maximum subgraph of $P(n, 2)$.

Proof. Assume that H contains $\lceil \frac{k}{2} \rceil$ outer rim vertices and $\lfloor \frac{k}{2} \rfloor$ inner rim vertices. Then, the number of edges induced by outer rim vertices is $\lceil \frac{k}{2} \rceil - 1$ and that of inner rim vertices is $\lfloor \frac{k}{2} \rfloor - 2$. Also the number of spokes induced by H is $\lfloor \frac{k}{2} \rfloor$. Therefore $|E(H)| = \lceil \frac{k}{2} \rceil - 1 + \lfloor \frac{k}{2} \rfloor - 2 + \lfloor \frac{k}{2} \rfloor = k + \lfloor \frac{k}{2} \rfloor - 3$. The same argument is true if H contains $\lceil \frac{k}{2} \rceil$ number of inner rim vertices and $\lfloor \frac{k}{2} \rfloor$ number of outer rim vertices. By Theorem 2.2, H is a maximum subgraph of $P(n, 2)$.

3. Embedding Generalized Petersen Graph into Cycle of Triangle

Theorem 3.1. [11] *The exact wirelength of Petersen graph $P(n, 1)$ into cycle of triangle CT_n is given by $WL(P(n, 1), CT_n) = 4n$.*

Theorem 3.2. *The exact wirelength of Petersen graph $P(n, 2)$ into Cycle of Triangle CT_n is given by $WL(P(n, 2), CT_n) = 5n$.*

Proof. Label the vertices of $P(n, 2)$ using parallel labelling and label the vertices of CT_n using zigzag labelling as $\{0, 1, \dots, n - 1\}$.

Case (i) (n odd)

For $0 \leq i \leq n - 1$, let $X_i = \{(v_i, v_{i+1}), (v_i, u_i), (v_{\frac{n-1}{2}+i}, v_{\frac{n+1}{2}+i}), (v_{\frac{n+1}{2}+i}, u_{\frac{n-1}{2}+i})\}$ be

an edge cut of CT_n . Then for each i , $CT_n - X_i$ has two components XH_i^1, XH_i^2 which are consecutively labelled and let $G_{i_1} = f^{-1}(XH_i^1)$ and $G_{i_2} = f^{-1}(XH_i^2)$. Then by Theorem 2.3, G_{i_1} be the induced subgraph of $P(n, 2)$ on n vertices is the maximum subgraph. Therefore by Lemma 1.1, $EC_f(X_i)$ is minimum and using Theorem 2.2, $EC_f(X_i) = 3n - 2(n + \lfloor \frac{n}{2} \rfloor - 3) = 7$.

For $0 \leq i \leq n-1$, let $Y_i = \{(u_i, v_i), (u_i, v_{i+1})\}$ be an edge cut of CT_n . Then $CT_n - Y_i$ yields the components YH_i^1 which is single vertex and let $G_i = f^{-1}(YH_i^1)$. Clearly, G_i be the induced subgraph of $P(n, 2)$ is maximum. Therefore by Lemma 1.1, $EC_f(Y_i)$ is minimum and using Theorem 2.2, $EC_f(Y_i) = 3$.

Hence by Lemma 1.4, the wirelength of Petersen graph $P(n, 2)$ into cycle of triangle CT_n is

$$WL(P(n, 2), CT_n) = \frac{1}{2} \sum_{i=0}^{n-1} \{EC_f(X_i) + EC_f(Y_i)\} = 5n.$$

Case (ii) (n even)

For $0 \leq i \leq (n/2) - 1$, let $A_i = \{(v_i, v_{i+1}), (v_i, u_i), (v_{i+1}, u_i), (v_{(n/2)+i}, v_{(n/2)+i+1}), (v_{(n/2)+i}, u_{(n/2)+i}), (v_{(n/2)+i+1}, u_{(n/2)+i})\}$ be an edge cut of CT_n as shown in Figure 1. Then $CT_n - A_i$ yields 4 components $AH_i^l, 1 \leq l \leq 4$ which are consecutively labelled and $|AH_i^1| = n-1$, $|AH_i^2| = 1$, $|AH_i^3| = n-1$, and $|AH_i^4| = 1$. Let $G_{i_l} = f^{-1}(AH_i^l), 1 \leq l \leq 4$. By Theorem 2.3, G_{i_l} be the induced subgraph of $P(n, 2)$ is the maximum. Therefore by Lemma 1.2, $EC_f(A_i)$ is minimum and using Theorem 2.2, $EC_f(A_i) = 3(n-1) - 2((n-1) + (\lfloor \frac{n-2}{2} \rfloor) - 3) = 10$. By Lemma 1.3, the wirelength of Petersen graph $P(n, 2)$ into cycle of triangle CT_n is calculated by $WL(P(n, 2), CT_n) = \sum_{i=0}^{(n/2)-1} EC_f(A_i) = 5n$.

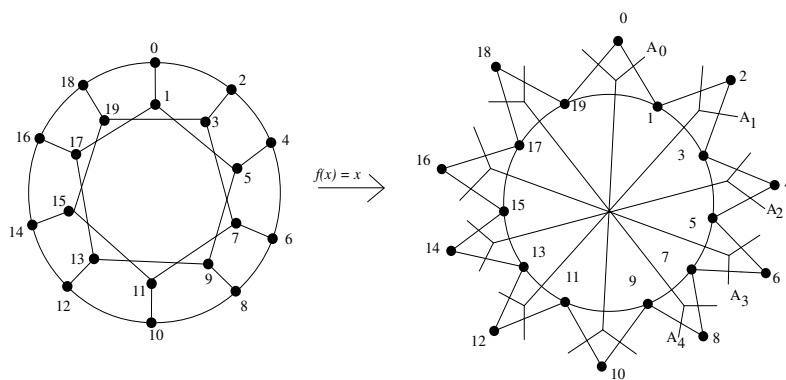


Figure 1: Embedding of Petersen Graph $P(10, 2)$ into CT_{20}

By extending the above arguments, we can easily arrive the following result.

Theorem 3.3. *The exact wirelength of Petersen graph $P(n, m)$, $1 \leq m \leq n - 1$ and $n \neq 2m$ into cycle of triangle CT_n is given by $WL(P(n, m), CT_n) = (m+3)n$.*

4. Conclusion

In this paper, we have found the exact wirelength of generalized Petersen graph $P(n, m)$ into cycle of triangle CT_n by effectively using generalized congestion and partition lemmas.

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